Q. Solve $\int_C \vec{F}\cdot d\vec{r}$ where (1)

$\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$

where C is from (0, 0, 0) to (1, 1, 1) along the path $x=t,\ y=t^2,\ z=t^3$.

Soln $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$

But $x=t,\ y=t^2,\ z=t^3$

$\Rightarrow \vec{F} = (3t^2+6t^2)\vec{i} - 14\,t^2.t^3\,\vec{j} + 20\,t.\,t^6\,\vec{k}$

$\Rightarrow \vec{F} = 9t^2\vec{i} - 14t^5\vec{j} + 20t^7\vec{k}$ ——(1)

Also $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$

$\Rightarrow$ ~~$d\vec{r} = \ldots$~~ $\Rightarrow d\vec{r} = dt\,\vec{i} + 2t\vec{j} + 3t^2\vec{k}$ ——(2)

From (1) and (2) we've

$\therefore \vec{F}\cdot d\vec{r} = 9t^2\,dt + 2t.(-14t^5) + 3t^2.20t^7$

$= 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$

Now $\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2\,dt - 28t^6\,dt + 60t^9\,dt)$

$= \left[\frac{9t^3}{3} - 28\frac{t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$

$= (3-4+6) = \underline{\underline{5}}$.

Q. Evaluate $\int \vec{F}\cdot d\vec{r}$ where $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ along the straight lines from $(0,0,0)$ to $(1,0,0)$, then to $(1,1,0)$ and then to $(1,1,1)$.

②

Soln From the question,

$$\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k} \quad \text{— (1)}$$

I Here C is from $(0,0,0)$ to $(1,0,0)$

i.e. $x = 0$ to 1 and $y=0,\ z=0 \Rightarrow dy=0, dz=0$

∴ (1) becomes

$$\vec{F} = 3x^2\vec{i} - 0 + 0 = 3x^2\vec{i} \quad \text{— (2)}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i}+0+0 = x\vec{i} \Rightarrow d\vec{r} = dx\,\vec{i} \quad \text{— (3)}$$

From (2) and (3)

$$\vec{F}\cdot d\vec{r} = 3x^2\,dx \Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} 3x^2\,dx$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \left[x^3\right]_0^1 = 1$$

II Here c is (1,0,0) to (1,1,0).

i.e $x = 1$, $y = 0$ to 1 and $z = 0$

$\Rightarrow dx = 0, \quad dz = 0.$

$\therefore \vec{F} = (3.1 + 6y)\vec{i} - 14y.0\vec{j} + 20.1.0\vec{k}$

$\vec{F} = (3+6y)\vec{i}$

Also, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \vec{i} + y\vec{j} + 0\vec{k} = \vec{i} + y\vec{j}$

$\Rightarrow d\vec{r} = 0 + dy\vec{j} = dy\,\vec{j}$

$\therefore \vec{F}\cdot d\vec{r} = (3+6y)\vec{i}\cdot\vec{j}\,dy = 0$

$$\therefore \int_{(1,0,0)}^{(1,1,0)} \vec{F}\cdot d\vec{r} = 0.$$

III Here c is from (1,1,0) (1,1,1).

i.e. $x = 1$, $y = 1$, and $z = 0$ to 1 $\Rightarrow dx = 0, dy = 0$

$\therefore \vec{F} = (3+6)\vec{i} - 14z\vec{j} + 20z^2\vec{k} = 9\vec{i} - 14z\vec{j} + 20z^2\vec{k}$

Also, $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} = dz\,\vec{k}$.

$\therefore \vec{F}\cdot d\vec{r} = 20z^2\,dz \Rightarrow \int_{(1,1,0)}^{(1,1,1)} \vec{F}\cdot d\vec{r} = \int_{z=0}^{1} 20z^2\,dz = \frac{20}{3}\left[z^3\right]_0^1$

$= \frac{20}{3}.$